Amalgamated Worksheet # 3

Various Artists

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1 Mike Hartglass

Unless otherwise stated, assume V is a finite dimensional complex vector space

1.) Do the following formulae define inner products on the given vector spaces? (here $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{C}^2$

$a.$) $V = \mathbb{C}^2$, $\langle x, y \rangle = x_1 y_1 + x_2 y_2$

$b.$) $V = \mathbb{C}^2$, $\langle x, y \rangle = x_1 \overline{y_1} + x_2 \overline{y_2}$

$c.$) $V = \mathbb{C}^2$, $\langle x, y \rangle = x_1 \overline{y_2} + x_2 \overline{y_1}$

$d.$) $V = \mathcal{P}^2(\mathbb{C})$, $\langle p, q \rangle = p(0)\overline{q(0)} + p(\sqrt{2})\overline{q(\sqrt{2})} + p(\pi)\overline{q(\pi)}$

2.) Suppose u and v are nonzero vectors in an inner product space v.

$a.$) Define

$$y = \frac{\langle v, w \rangle}{\langle w, w \rangle} w \text{ and } z = v - \frac{\langle v, w \rangle}{\langle w, w \rangle} w.$$

Show that $v = y + z$, $y \in \text{span}(w)$, and z is orthogonal to every vector in span(w).
$b.$) Draw a picture of this in $\mathbb{R}^2$ for $w = (1, 0)$ and $v = (1, 1)$.

3.) Suppose $(e_1, ..., e_n)$ is an orthonormal basis for a vector space V, and let $x = c_1 e_1 + \cdots c_n e_n$. Find a formula for the $c_i' s$.

4.) $a.$) Suppose x and y are orthogonal vectors in an inner product space V. Prove that

$$\|x + y\|^2 = \|x^2\| + \|y_2\|$$

b.) Suppose x and y are vectors in an inner product space V. Prove that

$$\|x + ay\| \geq \|x\| \text{ for all } a \in \mathbb{F} \text{ if and only if } \langle x, y \rangle = 0.$$

Draw a picture of this in $\mathbb{R}^2$.

2 Peyam Tabrizian

Problem 1:

Suppose $\langle , \rangle$ is an inner product on W, and $T : V \to W$ is injective. Show that:

$$(u, v) := \langle T(u), T(v) \rangle$$

is an inner product on V.

Problem 2:

Show that if $v_1, \cdots, v_k$ are nonzero orthogonal vectors, then $(v_1, \cdots, v_k)$ is linearly independent.

Problem 3:

Suppose $T \in \mathcal{L}(V)$ is self-adjoint. Show that every eigenvalue of T is real.

Problem 4:

Show that if T is normal, then $Nul(T^*) = Nul(T)$

Problem 5:

Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and U is a subspace of V.
Show that U is invariant under T if and only if $U^\perp$ is invariant under T^*

Problem 6:

(if time permits) Suppose V is finite-dimensional and U is a subspace of V.
Show that $V = U \oplus U^\perp$

Problem 7:

(if time permits) Let $(v_1, \cdots, v_n)$ be an orthonormal basis of V and suppose the matrix of $T \in \mathcal{L}(V)$ is A. What is the matrix of T^* with respect to that same basis?

3 Daniel Sparks

1

Let $U = \text{Span}(u_1, \cdots, u_m)$ and $W = \text{Span}(w_1, \cdots, w_k)$ be two subspaces of an inner product space V. Suppose that for each $1 \leq i \leq m, 1 \leq j \leq k$ that $\langle u_i, w_j \rangle = 0$. Prove that $U \perp W$.

2

Let $P : V \to V$ be a projection onto the subspace U. That is, suppose that $P^2 = P$ and $P(V) = U$. Prove that P is self-adjoint if and only if P is an orthogonal projection, that is, if and only if $\text{null}(P) \perp \text{range(P)}$.